Lecture 14 - Capacitors

A Puzzle...

Gravity Screen

Insulators are often thought of as "electrical screens," since they block out all external electric fields. For example, if neutral objects are kept inside of a conducting shell (of any shape), the electric field from any charge distribution outside the conductor will not be felt inside this cavity (by Gauss's Law).

<u>Example</u>
What is wrong with the idea of a gravity screen, something that will "block" gravity the way a metal sheet seems to "block" the electric field?
Hint: Think about the difference between the gravitational source and electrical sources.

<u>Solution</u>
Gravity cannot be blocked for two very important reasons:

- We need opposite-signed charges to block an electric field, so we would particles with negative mass (which to date have not been discovered).
- In gravitation (unlike in electricity), like charges are attracted to each other. So if we had a fixed point mass located outside a spherical shell, we could glue some negative mass on the near side of the shell in such a configuration so as to cancel the gravitational effects of this point mass inside of the shell. (We have to glue these masses, since otherwise they would be attracted towards each other and stop blocking the gravitational field.) But if this point mass was then moved, then we would need to manually update our mass configuration; this would significantly more cumbersome than the electrical case. □

What's in a Candle?

Have you ever seen a candle's flame split in two? This video demonstrates how a flame is comprised of positively and negatively charged ions, together with some dramatic repercussions that occur when you stick a flame between a parallel plate capacitor.

Theory

Capacitance

The capacitance of two conductors is always calculated using the same simple recipe:

1. Draw your two conductors of interest in the absence of any other charges or electric fields
2. Give one of the conductors a charge $+Q$ and the other a charge $-Q$ (it does not matter which one is positive)
3. Calculate the (positive) voltage difference V between the two conductors
4. The <u>*capacitance*</u> C between two conductors is defined as

$$C \equiv \frac{Q}{V} \tag{1}$$

Some more notes:

- We define the capacitance of a *single* conductor by assuming that the second conductor is a sphere with infinite radius. In other words, V is the potential difference between the surface of the conductor in the problem and infinity
- Capacitance is a property of the *geometry* of conductors. In other words, even if in your charge configuration the two conductors have a charge Q_1 and Q_2, if you compute their capacitance you always start off by assuming that they are neutral and then assigning a charge $+Q$ to one and $-Q$ to the other
- We will cover exactly how the charge Q gets transferred from one conductor to the other next week. As a sneak peak, you can imagine connecting a battery between the two neutral conductors which will transfer the charge Q from one to the other. We then disconnect this battery and calculate the capacitance of the resulting setup

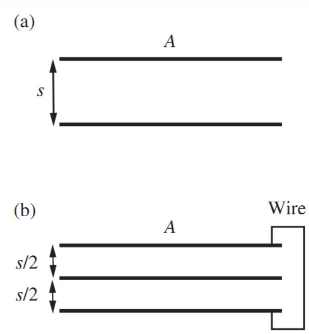
Complementary Section: Two Concentric Spheres

Problems

Inserting a Plate

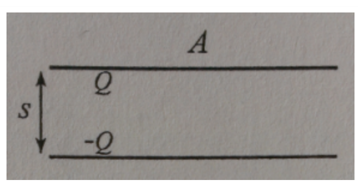
<u>Example</u>

If the capacitance in Panel (a) below is C, what is the capacitance in Panel (b), where a third plate is inserted and the outer plates are connected by a wire?



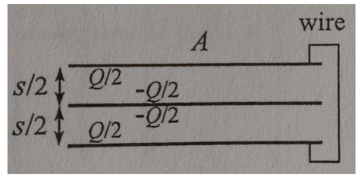
<u>Solution</u>

When we put charge $\pm Q$ on the two capacitors in Panel (a), it will spread out uniformly on the inner surfaces of both conductors. Since the electric field inside both conductors is zero, the Uniqueness Theorem guarantees that this is the only way the charge can be distributed.



If we define the surface charge $\sigma = \frac{Q}{A}$, the electric field inside will be $E = \frac{\sigma}{\epsilon_0} = \frac{Q}{A\,\epsilon_0}$ so the potential difference between the two conductors will be $V = E\,s = \frac{Q\,s}{A\,\epsilon_0}$. Therefore $C = \frac{Q}{V} = \frac{A\,\epsilon_0}{s}$.

Once we insert a new plate, we can use the symmetry of the problem to predict that $\frac{Q}{2}$ of charge will spread evenly on the inner surfaces of the two outer plates while $-\frac{Q}{2}$ of charge will spread evenly on both sides of the inner plate. Again, since $E = 0$ in every conductor, and both outer conductors are at the same potential, the Uniqueness Theorem guarantees that this will be the final charge distribution.



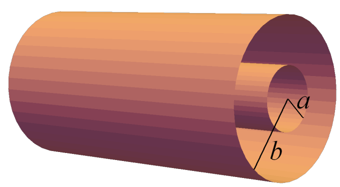
Defining $\tilde{\sigma} = \frac{Q}{2A}$, the electric field between the middle and top plate will be $\tilde{E} = \frac{\tilde{\sigma}}{\epsilon_0} = \frac{Q}{2A\epsilon_0}$ so the potential difference between those two plates equals $\tilde{V} = \tilde{E}\,\frac{s}{2} = \frac{Q s}{4A\epsilon_0}$. Therefore $\tilde{C} = \frac{Q}{\tilde{V}} = \frac{4A\epsilon_0}{s} = 4\,C$.

In the more general case where the middle plate is a fraction f of the distance from one of the outside plates to the other, you can show that the capacitance is $\frac{C}{f(1-f)}$. This correctly equals $4\,C$ when $f = \frac{1}{2}$. It is minimum when $f = \frac{1}{2}$ and goes to infinity as f goes to 0 or 1. □

Coaxial Capacitor

Example

A capacitor consists of two coaxial cylinders of length L, with inner and outer radii a and b. Assume $L \gg b - a$, so that end corrections may be neglected. Show that the capacitance is $C = \frac{2\pi L\epsilon_0}{\text{Log}[b/a]}$. Verify that if the gap between the cylinders, $b - a$, is very small compared with the radius, this result reduces to one that could have been obtained by using the formula for the parallel-plate capacitor.



Solutions

Assume that the inner cylinder has a charge Q while the outer cylinder has a charge $-Q$. Using Gauss's Law, the electric field at a radius $a < r < b$ is given by $\vec{E} = \frac{Q}{2\pi r L\epsilon_0}\hat{r}$. The (absolute value of the) potential between the two plates is given by the radial line integral going from $r = a$ to $r = b$,

$$\begin{aligned} V &= \int_a^b \vec{E}\cdot d\vec{s} \\ &= \int_a^b \left(\frac{Q}{2\pi r L\epsilon_0}\hat{r}\right)\cdot(\hat{r}\,dr) \\ &= \int_a^b \frac{Q}{2\pi r L\epsilon_0}\,dr \\ &= \frac{Q}{2\pi L\epsilon_0}\,\text{Log}\left[\frac{b}{a}\right] \end{aligned} \quad (6)$$

Therefore, the capacitance is given by

$$C = \frac{Q}{V} = \frac{2\pi L\epsilon_0}{\text{Log}[b/a]} \quad (7)$$

In the limit of a small gap $s = b - a \ll a$, we can Taylor expand the logarithm as

$$\text{Log}\left[\frac{b}{a}\right] = \text{Log}\left[\frac{a+s}{a}\right] = \text{Log}\left[1 + \frac{s}{a}\right] \approx \frac{s}{a} + O\left[\frac{s}{a}\right]^2 \quad (8)$$

Noting that the area of either cylinder in this limit equals $A = 2\pi a L$, the capacitance reduces to

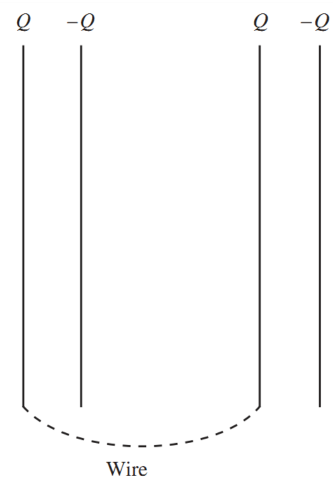
$$C = \frac{2\pi L \epsilon_0}{s/a} = \frac{A \epsilon_0}{s} \tag{9}$$

which is identical to the capacitance of parallel plates. This must be the case, since in the limit $s \ll a$, we can think of the two cylinders as many parallel plate capacitors connected in parallel (see Problem 3.18). □

Capacitor Paradox

Example

Two capacitors with the same capacitance C and charge Q are placed next to each other. The two positive plates are then connected by a wire. Will charge flow in the wire? Consider two possible scenarios:



(A) Before the plates are connected, the potential differences of the two capacitors are the same (because Q and C are the same). So the potentials of the two positive plates are equal. Therefore, no charge will flow in the wire when the plates are connected.

(B) Number the plates 1 through 4, from left to right. Before the plates are connected, there is zero electric field in the region between the capacitors, so plate 3 must be at the same potential as plate 2. But plate 2 is at a lower potential than plate 1. Therefore, plate 3 is at a lower potential than plate 1, so charge will flow in the wire when the plates are connected.

Which reasoning is correct, and what is wrong with the wrong reasoning?

Solution

Reasoning (B) is correct. Plate 3 is indeed at a lower potential than plate 1, so charge will flow. The error in the first reasoning is encompassed in the word, "So." Although it is true that the potential differences of the two capacitors are the same, this does not imply that the potentials of the two positive plates are equal. If we arbitrarily assign zero potential to plate 1, and if the common potential difference is ϕ, then the potentials of the fours plates are, from left to right, $0, -\phi, -\phi, -2\phi$. No matter where we define the zero of potential, the potential of the leftmost plate is ϕ larger than the potential of the third plate, and 2ϕ larger than the potential of the rightmost plate. □

A Four-Plate Capacitor

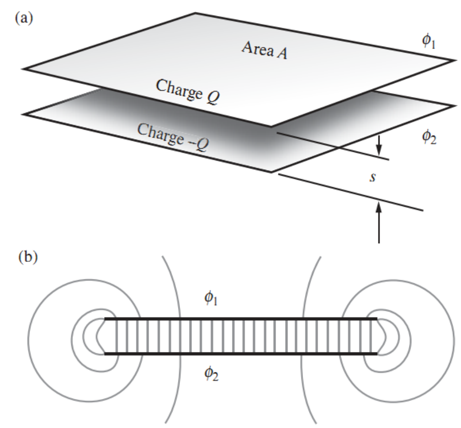
A 2 *N*-Plate Capacitor

A Three-Shell Capacitor

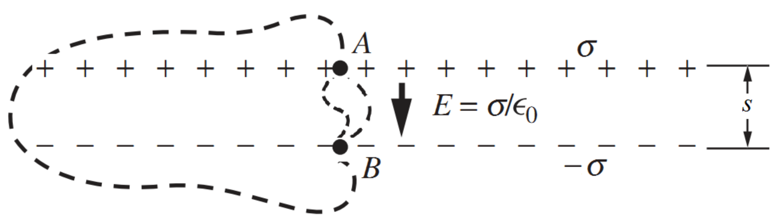
Advanced Section: Edge Effects

Let us return to the canonical example of a parallel plate capacitor. As discussed in Equation (3.13), the total charge on the top plate of this capacitor is given by

$$Q = A\,\frac{\epsilon_0(\phi_1 - \phi_2)}{s} \quad \text{(neglecting edge effects)} \tag{28}$$



From this point on, it is important to keep in the back of your mind that our dealings with the parallel plate capacitor will *almost* always involve neglecting edge effects. This "almost" is especially important because we will occasionally look at phenomena that are *entirely caused by edge effects*. At such points, it is sometimes difficult to separate out what we have learned that is hinges on neglecting edge effects and what is generally true. As a simple example, consider the potential difference between the middle points on the top and bottom conductors shown above.



Far away from the edges, the electric field inside the capacitor will be uniform, so that the potential difference along any path from A to B (including the straight line path between them) must equal $\frac{\sigma}{\epsilon_0} s$. So what about an exterior path between A and B? If we neglect edge effects, then the electric field outside the capacitor is zero, and so we would *incorrectly* conclude that the potential difference between A and B is 0. But we know one of these results must be wrong, since the potential between any two points is independent of the path taken between these two points, and indeed it is this latter argument (which neglects edge effects) which is incorrect.

Another interesting example is considered in "Advanced Section: Conductor in a Capacitor" below, where we discuss how a conducting slab is sucked into a parallel plate capacitor entirely due to edge effects.

Advanced Section: Conductor in a Capacitor

Mathematica Initialization